

# Aviation Formulary V1.37

By Ed Williams

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### Introduction

This introduction is written for pilots (and others) who are interested in great circle navigation and would like to know how to compute courses, headings and other quantities of interest. These formulae can be programmed into your calculator or spreadsheet. I'll attempt to include enough information that those familiar with plane trigonometry can derive additional results if required.

It is a well known that the shortest distance between two points is a straight line. However anyone attempting to fly from Los Angeles to New York on the straight line connecting them would have to dig a very substantial tunnel first. The shortest distance, *following the earth's surface* lies vertically above the aforementioned straight line route. This route can be constructed by slicing the earth in half with an imaginary plane through LAX and JFK. This plane cuts the (assumed spherical) earth in a circular arc connecting the two points, called a *great circle*. Only planes through the center of the earth give rise to great circles. Any plane will cut a sphere in a circle, but the resulting little circles are not the shortest distance between the points they connect. A little thought will show that lines of longitude (meridians) are great circles, but lines of latitude, with the exception of the equator, are not.

I will assume the reader is familiar with latitude and longitude as a means of designating locations on the earth's surface. For the convenience of North Americans I will take North latitudes and West longitudes as positive and South and East negative. The longitude is the opposite of the usual mathematical convention. True course is defined as usual, as the angle between the course line and the local meridian measured clockwise.

The first important fact to realise is that in general a great circle route has a true course that varies from point to point. For instance the great circle route between two points of equal (non-zero) latitude does not follow the line of latitude in an E-W direction, but arcs towards the pole. It is possible to fly between two points using an unvarying true course, but in general the resulting route differs from the great circle route and is called a *rhumb* line. Unlike a great circle which encircles the earth, a pilot flying a rhumb line would spiral indefinitely poleward.

Natural questions are to seek the great circle distance between two specified points and true course at points along the route. The required spherical trigonometric formulae are greatly simplified if angles and distances are measured in the appropriate natural units, which are both radians! A radian, by definition, is the angle subtended by a circular arc of unit length and unit radius. Since the length of a complete circular arc of unit radius is  $2\pi$ , the conversion is 360 degrees equals  $2\pi$  radians, or:

```
angle_radians=(pi/180)*angle_degrees
angle_degrees=(180/pi)*angle_radians
```

Great circle distance can be likewise be expressed in radians by defining the distance to be the angle subtended by the arc at the center of the earth. Since by definition, one nautical mile subtends one minute ( $=1/60$  degree) of arc, we have:

```
distance_radians=(pi/(180*60))*distance_nm
distance_nm=((180*60)/pi)*distance_radians
```

In all subsequent formulae all distances and angles, such as latitudes, longitudes and true courses will be assumed to be given in radians, greatly simplifying them, and in applications the above formulae and their inverses are necessary to convert back and forth between natural and practical units. Examples of this process are given later.

Note: the nautical mile is currently defined to be 1852 meters - which to be consistent with its historical definition implies the earth's radius to be  $1.852 * (180*60/\pi) = 6366.71$  km, which indeed lies between the currently accepted ( WGS84) equatorial and polar radii of 6378.137 and 6356.752 km, respectively. Other choices of the earth's radius in this range are consistent with the spherical approximation and may for some specialized purposes be preferred.

For instance, the former FAI standard for aviation records used the "FAI sphere" with radius 6371.0 km.

To use a different spherical radius, use

```
distance_km = radius_km * distance_radians
distance_radians = distance_km/radius_km
```

to convert between angular and distance units.

A sample implementation of many of these formulae in the form of an Excel spreadsheet can be found [here](#). The formulae are in VBA macros, for readability, so you will need to enable macros to have them work. If you are unable to open the spreadsheet with macros disabled (to check for viruses) etc, then you may need to patch your Excel. Try <http://office.microsoft.com/downloadaddetails/Xl8p9pkg.htm> for Excel 97 SR-2.

*If you decide to program up these formulae, you'd be well-advised to look at the [implementation notes](#) and check your results against the [worked examples](#) and spreadsheets.*

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## Some great circle formulae:

### Distance between points

The great circle distance  $d$  between two points with coordinates  $\{\text{lat1}, \text{lon1}\}$  and  $\{\text{lat2}, \text{lon2}\}$  is given by:

```
d=acos(sin(lat1)*sin(lat2)+cos(lat1)*cos(lat2)*cos(lon1-lon2))
```

A mathematically equivalent formula, which is less subject to rounding error for short distances is:

```
d=2*asin(sqrt((sin((lat1-lat2)/2))^2 +
              cos(lat1)*cos(lat2)*(sin((lon1-lon2)/2))^2))
```

---

### Course between points

We obtain the initial course,  $tc1$ , (at point 1) from point 1 to point 2 by the following. The formula fails if the initial point is a pole. We can special case this with:

```
IF (cos(lat1) < EPS)    // EPS a small number ~ machine precision
  IF (lat1 > 0)
    tc1= pi             // starting from N pole
  ELSE
    tc1= 2*pi           // starting from S pole
  ENDIF
ENDIF
```

For starting points other than the poles:

```
IF sin(lon2-lon1)<0
  tc1=acos((sin(lat2)-sin(lat1)*cos(d))/(sin(d)*cos(lat1)))
ELSE
  tc1=2*pi-acos((sin(lat2)-sin(lat1)*cos(d))/(sin(d)*cos(lat1)))
ENDIF
```

An alternative formula, not requiring the pre-computation of d, the distance between the points, is:

```
tc1=mod(atan2(sin(lon1-lon2)*cos(lat2),
  cos(lat1)*sin(lat2)-sin(lat1)*cos(lat2)*cos(lon1-lon2)), 2*pi)
```

---

### Latitude of point on GC

Intermediate points {lat,lon} lie on the great circle connecting points 1 and 2 when:

```
lat=atan((sin(lat1)*cos(lat2)*sin(lon-lon2)
  -sin(lat2)*cos(lat1)*sin(lon-lon1))/(cos(lat1)*cos(lat2)*sin(lon1-lon2))
```

(not applicable for meridians. i.e if  $\sin(\text{lon1}-\text{lon2})=0$ )

---

### Lat/lon given radial and distance

A point {lat,lon} is a distance d out on the tc radial from point 1 if:

```
lat=asin(sin(lat1)*cos(d)+cos(lat1)*sin(d)*cos(tc))
IF (cos(lat)=0)
  lon=lon1 // endpoint a pole
ELSE
  lon=mod(lon1-asin(sin(tc)*sin(d)/cos(lat))+pi,2*pi)-pi
ENDIF
```

This algorithm is limited to distances such that  $d_{lon} < \pi/2$ , i.e those that extend around less than one quarter of the circumference of the earth in longitude. A completely general, but more complicated algorithm is necessary if greater distances are allowed:

```
lat =asin(sin(lat1)*cos(d)+cos(lat1)*sin(d)*cos(tc))
dlon=atan2(sin(tc)*sin(d)*cos(lat1),cos(d)-sin(lat1)*sin(lat))
lon=mod(lon1-dlon +pi,2*pi )-pi
```

---

### Intersecting radials

Now how to compute the latitude, lat3, and longitude, lon3 of an intersection formed by the crs13 true bearing from point 1 and the crs23 true bearing from point 2:

```
dst12=2*asin(sqrt((sin((lat1-lat2)/2))^2+
  cos(lat1)*cos(lat2)*sin((lon1-lon2)/2)^2))
IF sin(lon2-lon1)<0
  crs12=acos((sin(lat2)-sin(lat1)*cos(dst12))/(sin(dst12)*cos(lat1)))
```

```

ELSE
    crs12=2.*pi-acos((sin(lat2)-sin(lat1)*cos(dst12))/(sin(dst12)*cos(lat1)))
ENDIF
IF sin(lon1-lon2)<0
    crs21=acos((sin(lat1)-sin(lat2)*cos(dst12))/(sin(dst12)*cos(lat2)))
ELSE
    crs21=2.*pi-acos((sin(lat1)-sin(lat2)*cos(dst12))/(sin(dst12)*cos(lat2)))
ENDIF
ang1=mod(crs13-crs12+pi,2.*pi)-pi
ang2=mod(crs21-crs23+pi,2.*pi)-pi
IF (sin(ang1)*sin(ang2)<=sqrt(TOL))
    "no intersection exists"
ELSE
    ang1=abs(ang1)
    ang2=abs(ang2)
    ang3=acos(-cos(ang1)*cos(ang2)+sin(ang1)*sin(ang2)*cos(dst12))
    dst13=asin(sin(ang2)*sin(dst12)/sin(ang3))
    lat3=asin(sin(lat1)*cos(dst13)+cos(lat1)*sin(dst13)*cos(crs13))
    lon3=mod(lon1-asin(sin(crs13)*sin(dst13)/cos(lat3))+pi,2.*pi)-pi
ENDIF

```

TOL is a small number of order machine precision.  $10^{-15}$  would be OK for standard double precision arithmetic.

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## Intersections of two great circles

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### Clairaut's formula:

This relates the latitude (lat) and true course (tc) along any great circle, namely:  $\sin(tc) \cdot \cos(lat) = \text{constant}$ . That is, for any two points on the GC:

$$\sin(tc1) \cdot \cos(lat1) = \sin(tc2) \cdot \cos(lat2)$$

Since at the highest latitude (latmx) reached the tc must be 90/270, we also have:

$$latmx = \arccos(\sin(tc) \cdot \cos(lat))$$

where lat and tc are the latitude and true course at *any* point on the great circle.

---

### Crossing parallels:

Any given great circle (excepting one over the poles) crosses each meridian once and only once. However, any given great circle has a maximum latitude reached at its apex. It crosses lower latitudes twice and higher latitudes never. Thus the algorithm for finding the longitudes at which a given great circle crosses a given parallel is a little more complex.

Suppose a great circle passes through (lat1,lon1) and (lat2,lon2). It crosses the parallel lat3 at longitudes lon3\_1 and lon3\_2 given by:

$$\begin{aligned}
 l12 &= lon1 - lon2 \\
 A &= \sin(lat1) \cdot \cos(lat2) \cdot \cos(lat3) \cdot \sin(l12) \\
 B &= \sin(lat1) \cdot \cos(lat2) \cdot \cos(lat3) \cdot \cos(l12) - \cos(lat1) \cdot \sin(lat2) \cdot \cos(lat3) \\
 C &= \cos(lat1) \cdot \cos(lat2) \cdot \sin(lat3) \cdot \sin(l12)
 \end{aligned}$$

```

lon = atan2(B,A)                                ( atan2(y,x) convention)
IF (C > sqrt(A^2 + B^2))
    "no crossing"
ELSE
    dlon = acos(C/sqrt(A^2+B^2))
    lon3_1=mod(lon1+dlon+lon+pi, 2*pi)-pi
    lon3_2=mod(lon1-dlon+lon+pi, 2*pi)-pi
ENDIF

```

---

### Intermediate points on a great circle

In previous sections we have found intermediate points on a great circle given either the crossing latitude or longitude. Here we find points (lat,lon) a given fraction of the distance (d) between them. Suppose the starting point is (lat1,lon1) and the final point (lat2,lon2) and we want the point a fraction f along the great circle route. f=0 is point 1. f=1 is point 2. The two points cannot be antipodal ( i.e. lat1+lat2=0 and abs(lon1-lon2)=pi) because then the route is undefined. The intermediate latitude and longitude is then given by:

```

A=sin((1-f)*d)/sin(d)
B=sin(f*d)/sin(d)
x = A*cos(lat1)*cos(lon1) + B*cos(lat2)*cos(lon2)
y = A*cos(lat1)*sin(lon1) + B*cos(lat2)*sin(lon2)
z = A*sin(lat1)           + B*sin(lat2)
lat=atan2(z,sqrt(x^2+y^2))
lon=atan2(y,x)

```

---

### Cross track error:

Suppose you are proceeding on a great circle route from A to B (course =crs\_AB) and end up at D, perhaps off course. You can calculate the course from A to D (crs\_AD) and the distance from A to D (dist\_AD) using the formulae above. In terms of these the cross track error, XTD, (distance off course) is given by

$$XTD = \text{asin}(\sin(\text{dist\_AD}) * \sin(\text{crs\_AD} - \text{crs\_AB}))$$

(positive XTD means right of course, negative means left)

The "along track distance", ATD, the distance from A along the course towards B to the point abeam D is given by:

$$ATD = \text{acos}(\cos(\text{dist\_AD}) / \cos(XTD))$$

For very short distances:

$$ATD = \text{asin}(\sqrt{(\sin(\text{dist\_AD}))^2 - (\sin(XTD))^2} / \cos(XTD))$$

is less susceptible to rounding error

---

### Implementation notes:

#### Notes on mathematical functions

Note: ^ denotes the exponentiation operator, sqrt is the square root function, acos the arc-cosine (or inverse cosine) function and asin is the arc-sine function. If asin or acos are unavailable they can be implemented using the atan2 function:

```
acos(x)=atan2(sqrt(1-x^2),x)
  acos returns a value in the range 0 <= acos <= pi
asin(x)=atan2(x,sqrt(1-x^2))
  asin returns a value in the range -pi/2 <= asin <= pi/2
```

Note: Here atan2 has the conventional (C) ordering of arguments, namely atan2(y,x). This is not universal, Excel for instance uses atan2(x,y), but it has asin and acos anyway. Be warned. It returns a value in the range  $-\pi < \text{atan2} \leq \pi$ .

Further note: if your calculator/programming language is so impoverished that only atan is available then use:

```
asin(x)=2*atan(x/(1+sqrt(1-x*x)))
acos(x)=2*atan(sqrt((1-x)/(1+x)))      x>=0
      =pi - 2*atan(sqrt((1+x)/(1-x)))  x<0

atan2(y,x)=atan(y/x)                  x>0
atan2(y,x)=atan(y/x)+pi                x<0, y>=0
atan2(y,x)=pi/2                        x=0, y>0
atan2(y,x)=atan(y/x)-pi                 x<0, y<0
atan2(y,x)=-pi/2                       x=0, y<0
atan2(0,0) is undefined and should give an error.
```

Another potential implementation problem is that the arguments of asin and/or acos may, because of rounding error, exceed one in magnitude. With perfect arithmetic this can't happen. You may need to use "safe" versions of asin and acos on the lines of:

```
asin_safe(x)=asin(max(-1,min(x,1)))
acos_safe(x)=acos(max(-1,min(x,1)))
```

Note on the mod function. This appears to be implemented differently in different languages. Mod(y,x) is the remainder on dividing y by x and always lies in the range  $0 \leq \text{mod} < x$ . For instance: mod(2.3,2.)=0.3 and mod(-2.3,2.)=1.7

If you have a floor function (int in Excel), that returns floor(x)= "largest integer less than x" e.g. floor(-2.3)=-3 and floor(2.3)=2

$$\text{mod}(y, x) = y - x * \text{floor}(y/x)$$

The following should work in the absence of a floor function- regardless of whether "int" truncates or rounds downward:

```
mod=y - x * int(y/x)
if ( mod < 0) mod = mod + x
```

### Sign Convention

As stated in the introduction, North latitudes and *West* longitudes are treated as positive, and South latitudes and East longitudes negative. It's easier to go with the flow, but if you prefer another convention you can change the signs in the formulae.

## Worked Examples:

Suppose point 1 is LAX: (33deg 57min N, 118deg 24min W)

Suppose point 2 is JFK: (40deg 38min N, 73deg 47min W)

In radians LAX is

$$(33+57/60)*\pi/180=0.592539, \quad (118+24/60)*\pi/180=2.066470$$

and JFK is

$$(0.709186, 1.287762)$$

The distance from LAX to JFK is

$$\begin{aligned} d &= \arccos(\sin(\text{lat1}) * \sin(\text{lat2}) + \cos(\text{lat1}) * \cos(\text{lat2}) * \cos(\text{lon1} - \text{lon2})) \\ &= \arccos(\sin(0.592539) * \sin(0.709186) + \\ &\quad \cos(0.592539) * \cos(0.709186) * \cos(0.778708)) \\ &= \arccos(0.811790) \\ &= 0.623585 \text{ radians} \\ &= 0.623585 * 180 * 60 / \pi = 2144 \text{ nm} \end{aligned}$$

The initial true course out of LAX is:

$$\begin{aligned} \sin(-0.778708) &= -0.702 < 0 \text{ so} \\ \text{tc1} &= \arccos((\sin(\text{lat2}) - \sin(\text{lat1}) * \cos(d)) / (\sin(d) * \cos(\text{lat1}))) \\ &= \arccos((\sin(0.709186) - \sin(0.592539) * \cos(0.623585)) / \\ &\quad (\sin(0.623585) * \cos(0.592539))) \\ &= \arccos(0.408455) \\ &= 1.150035 \text{ radians} \\ &= 66 \text{ degrees} \end{aligned}$$

An enroute waypoint 100nm from LAX on the 66 degree radial (100nm along the GC to JFK) has lat and long given by:

$$\begin{aligned} 100 \text{ nm} &= 100 * \pi / (180 * 60) = 0.0290888 \text{ radians} \\ \text{lat} &= \arcsin(\sin(\text{lat1}) * \cos(d) + \cos(\text{lat1}) * \sin(d) * \cos(\text{tc})) \\ &= \arcsin(\sin(0.592539) * \cos(0.0290888) \\ &\quad + \cos(0.592539) * \sin(0.0290888) * \cos(1.150035)) \\ &= \arcsin(0.568087) \\ &= 0.604180 \text{ radians} \\ &= 34 \text{ degrees } 37 \text{ min N} \\ \text{lon} &= \text{mod}(\text{lon1} - \arcsin(\sin(\text{tc}) * \sin(d) / \cos(\text{lat})) + \pi, 2 * \pi) - \pi \\ &= \text{mod}(2.066470 - \arcsin(\sin(1.150035) * \sin(0.0290888) / \cos(0.604180)) + \pi, 2 * \pi) - \pi \\ &= \text{mod}(2.034206 + \pi, 2 * \pi) - \pi \text{ radians} \\ &= 2.034206 \text{ radians} \\ &= 116 \text{ degrees } 33 \text{ min W} \end{aligned}$$

The great circle route from LAX to JFK crosses the 111 degree W meridian at a latitude of:

$$(111 \text{ degrees} = 1.937315 \text{ radians})$$

$$\text{lat} = \arctan((\sin(\text{lat1}) * \cos(\text{lat2}) * \sin(\text{lon} - \text{lon2}))$$



```

-sin(lat2)*cos(lat1)*sin(lon-lon1))/(cos(lat1)*cos(lat2)*sin(lon1-lon2)))
= atan((sin(0.592539)*cos(0.709186)*sin(0.649553)
-sin(0.709186)*cos(0.592539)*sin(-0.129154))/(cos(0.592539)*cos(0.709186)
*sin(0.778708)))
= atan(0.737110)
= 0.635200radians
= 36 degrees 24min

```

### Cross track error

Suppose enroute from JFK to LAX you find yourself at (D) N34:30 W116:30, which in radians is (0.6021386,2.033309) (See earlier for LAX, JFK coordinates and course)

From LAX to D the distance is:

```

dist_AD = acos(sin(0.592539)*sin(0.6021386)+
cos(0.592539)*cos(0.6021386)*cos(2.066470-2.033309))
= 0.02905 radians (99.8665 nm)

```

From LAX to D the course is:

```

crs_AD = acos((sin(0.6021386)-sin(0.592539)*cos(0.02905))/
(sin(0.02905)*cos(0.592539)))
= 1.22473 radians (70.17 degrees)

```

At point D the cross track error is:

```

xtd = asin(sin(0.02905)*sin(1.22473-1.15003))
= 0.00216747 radians
= 0.00216747*180*60/pi =7.4512 nm right of course

atd = acos(cos(0.02905)/cos(0.00216747))
= 0.0289691 radians
= 0.0289691*180*60/pi = 99.588 nm along course

```

Example of an intersection calc (briefly):

Let point 1 be REO (42.60N,117.866W)=(0.74351,2.05715) rad  
Let point 2 be BKE (44.84N,117.806W)=(0.782606,2.056103) rad

The 51 degree (=0.890118rad) bearing from REO intersects with 137 degree (=2.391101rad) from BKE at (lat3,lon3):

Then:

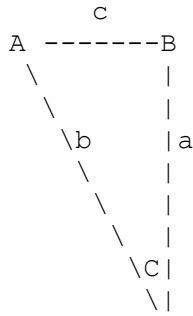
```

dst12=0.039103
crs12=0.018996
crs21=3.161312
ang1=0.871122
ang2=0.770211
ang3=1.500667
dst13=0.02729
dst23=0.029986
lat3=0.760473      =43.5N
lon3=2.027876      =116.2W  at BOI!

```

## Some general spherical triangle formulae.

A spherical triangle is one whose sides are all great circular arcs. Let the sides have lengths  $a, b$  and  $c$  radians, and the opposite angles be  $A, B$  and  $C$  radians.



(The angle at B is not necessarily a right angle)

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)}$$

$$\begin{aligned}\cos(a) &= \cos(b) \cos(c) + \sin(b) \sin(c) \cos(A) \\ \cos(b) &= \cos(c) \cos(a) + \sin(c) \sin(a) \cos(B) \\ \cos(c) &= \cos(a) \cos(b) + \sin(a) \sin(b) \cos(C)\end{aligned}$$

$$\begin{aligned}\cos(A) &= -\cos(B) \cos(C) + \sin(B) \sin(C) \cos(a) \\ \cos(B) &= -\cos(C) \cos(A) + \sin(C) \sin(A) \cos(b) \\ \cos(C) &= -\cos(A) \cos(B) + \sin(A) \sin(B) \cos(c)\end{aligned}$$

Some useful consequences of these are:

$$\begin{aligned}\tan(A) &= \sin(B) \sin(a) / (\sin(c) \cos(a) - \cos(B) \cos(c) \sin(a)) \\ \tan(B) &= \sin(C) \sin(b) / (\sin(a) \cos(b) - \cos(C) \cos(a) \sin(b)) \\ \tan(C) &= \sin(A) \sin(c) / (\sin(b) \cos(c) - \cos(A) \cos(b) \sin(c)) \\ \tan(a) &= \sin(b) \sin(A) / (\sin(C) \cos(A) + \cos(b) \cos(C) \sin(A)) \\ \tan(b) &= \sin(c) \sin(B) / (\sin(A) \cos(B) + \cos(c) \cos(A) \sin(B)) \\ \tan(c) &= \sin(a) \sin(C) / (\sin(B) \cos(C) + \cos(a) \cos(B) \sin(C))\end{aligned}$$

Given *any* three of  $\{a, b, c, A, B, C\}$  the remaining sides and angles can be found using these formulae. To solve a spherical triangle (requiring  $0 < a, b, c, A, B, C < \pi$  to get rid of pathological cases):

```
Given {A,b,c}: // Two sides, included angle
a=acos(cos(b)*cos(c)+sin(b)*sin(c)*cos(A))
B=acos((cos(b) - cos(c)*cos(a))/(sin(c)*sin(a)))
C=acos((cos(c) - cos(a)*cos(b))/(sin(a)*sin(b)))
```

```
Given {a,B,C}: // Two angles, included side
A=acos(-cos(B)*cos(C)+sin(B)*sin(C)*cos(a))
b=atan2(sin(a)*sin(B)*sin(C),cos(B)+cos(C)*cos(A))
c=atan2(sin(a)*sin(B)*sin(C),cos(C)+cos(A)*cos(B))
```

```
Given {a,b,c}: // Three sides
A=acos((cos(a) - cos(b)*cos(c))/(sin(b)*sin(c)))
B=acos((cos(b) - cos(c)*cos(a))/(sin(c)*sin(a)))
C=acos((cos(c) - cos(a)*cos(b))/(sin(a)*sin(b)))
```

Given  $\{A, B, C\}$ : // Three angles (this has an infinity of solutions for plane triangles and so is potentially numerically inaccurate for small

```

spherical triangles)
delta=(A+B+C-pi)/2
a=2*asin(sqrt(sin(delta)*sin(A-delta)/(sin(B)*sin(C))))
b=2*asin(sqrt(sin(delta)*sin(B-delta)/(sin(C)*sin(A))))
c=2*asin(sqrt(sin(delta)*sin(C-delta)/(sin(A)*sin(B))))

Given {A,a,b}: // Two sides, non-included angle
x=sin(A)*sin(b)/sin(a)
if (x=1) {
    B=pi/2 // One spherical triangle exists
} else if (x < 1) {
    B= asin(x) and pi-asin(x) // Two triangles exist
} else{
    // No triangles exist
}
For each triangle
c=mod(2*atan2(cos((A+B)/2)*sin((a+b)/2),cos((A-B)/2)*cos((a+b)/2)),2*pi)
C=mod(2*atan2(cos((a-b)/2)*cos((A+B)/2),cos((a+b)/2)*sin((A+B)/2)),2*pi)

Given {a,A,B}: // Two angles, non-included side
x=sin(a)*sin(B)/sin(A)
if (x=1) {
    b=pi/2 // One spherical triangle exists
} else if (x < 1) {
    b=asin(x) and pi-asin(x) // Two triangles exist
} else{
    // No triangles exist
}
For each triangle
c=mod(2*atan2(cos((A+B)/2)*sin((a+b)/2),cos((A-B)/2)*cos((a+b)/2)),2*pi)
C=mod(2*atan2(cos((a-b)/2)*cos((A+B)/2),cos((a+b)/2)*sin((A+B)/2)),2*pi)

```

For a spherical triangle the sum of the interior angles  $A+B+C$  is not  $\pi$  (180 degrees) but greater. The difference is called the spherical excess  $E$ , defined as  $E=A+B+C-\pi$ .

In terms of which the surface area enclosed by a spherical triangle is given by

$$\text{Area} = E \cdot R^2$$

In terms of the sides:

$$E = 4 \cdot \text{atan}(\sqrt{\tan(s/2) \cdot \tan((s-a)/2) \cdot \tan((s-b)/2) \cdot \tan((s-c)/2)})$$

where

$$s = (a+b+c)/2$$

analogous to Heron's formula for a plane triangle.

Some other formulae that may occasionally be useful are:

$$\begin{aligned}
 \sin(A/2) &= \sqrt{(\sin(s-b) \cdot \sin(s-c)) / (\sin(b) \cdot \sin(c))} \\
 \cos(A/2) &= \sqrt{(\sin(s) \cdot \sin(s-a)) / (\sin(b) \cdot \sin(c))} \\
 \tan(A/2) &= \sin((b-c)/2) / (\sin((b+c)/2) \cdot \tan((B-C)/2)) \\
 &= \cos((b-c)/2) / (\cos((b+c)/2) \cdot \tan((B+C)/2)) \\
 \tan(a/2) &= \cos((B+C)/2) \cdot \tan((b+c)/2) / \cos((B-C)/2) \\
 &= \sin((B+C)/2) \cdot \tan((b-c)/2) / \sin((B-C)/2) \\
 \tan((A-B)/2) &= \cot(C/2) \cdot \sin((a-b)/2) / \sin((a+b)/2)
 \end{aligned}$$

$$\begin{aligned}\tan((A+B)/2) &= \cot(C/2) * \cos((a-b)/2) / \cos((a+b)/2) \\ \sin(a) * \cos(B) &= \cos(b) * \sin(c) - \sin(b) * \cos(c) * \cos(A) \\ \cos(a) * \cos(C) &= \sin(a) * \cot(b) - \sin(C) * \cot(B)\end{aligned}$$

In any of these formulae, A, B and C can be interchanged, provided a, b and c change with them. ie a->b, b->c, c->a, A->B, B->C, C->A. In addition, the formulae hold if pi-a is written for A, pi-b for B and pi-c for C, etc. ie A->pi-a, B->pi-b, C->pi-c, a->pi-A, b->pi-B, c->pi-C

---

## Rhumb Line Navigation

Rhumb lines or loxodromes are tracks of constant true course. With the exception of meridians and the equator, they are not the same as great circles. They are not very useful approaching either pole, where they become tightly wound spirals. The formulae below fail if any point actually *is* a pole.

When two points (lat1,lon1), (lat2,lon2) are connected by a rhumb line with true course tc :

$$\begin{aligned}\text{lon2}-\text{lon1} &= -\tan(tc) * (\log((1+\sin(\text{lat2}))/\cos(\text{lat2})) - \\ &\quad \log((1+\sin(\text{lat1}))/\cos(\text{lat1}))) \\ &= -\tan(tc) * (\log((1+\tan(\text{lat2}/2))/(1-\tan(\text{lat2}/2))) - \\ &\quad \log((1+\tan(\text{lat1}/2))/(1-\tan(\text{lat1}/2)))) \\ &= -\tan(tc) * (\log(\tan(\text{lat2}/2+\pi/4)/\tan(\text{lat1}/2+\pi/4)))\end{aligned}$$

(logs are "natural" logarithms to the base e.)

The true course between the points is given by:

$$tc = \text{mod}(\text{atan2}(\text{lon1}-\text{lon2}, \log(\tan(\text{lat2}/2+\pi/4)/\tan(\text{lat1}/2+\pi/4))), 2*\pi)$$

The dist, d between the points is given by:

```
if (abs(lat2-lat1) < sqrt(TOL)) {
    q=cos(lat1)
} else {
    q= (lat2-lat1)/log(tan(lat2/2+pi/4)/tan(lat1/2+pi/4))
}
d=sqrt((lat2-lat1)^2+ q^2*(lon2-lon1)^2)
```

This formula fails if the rhumb line in question crosses the 180 E/W meridian. Allowing this as a possibility, the true course tc, and distance d, for the shortest rhumb line connecting two points is given by:

```
dlon_W=mod(lon2-lon1,2*pi)
dlon_E=mod(lon1-lon2,2*pi)
dphi=log(tan(lat2/2+pi/4)/tan(lat1/2+pi/4))
if (abs(lat2-lat1) < sqrt(TOL)) {
    q=cos(lat1)
} else {
    q= (lat2-lat1)/dphi
}
if (dlon_W < dlon_E) { // Westerly rhumb line is the shortest
    tc=mod(atan2(-dlon_W,dphi),2*pi)
    d= sqrt(q^2*dlon_W^2 + (lat2-lat1)^2)
} else {
    tc=mod(atan2(dlon_E,dphi),2*pi)
    d= sqrt(q^2*dlon_E^2 + (lat2-lat1)^2)
}
```

To find the lat/lon of a point on true course tc, distance d from (lat1,lon1) along a rhumbline (initial point cannot be a pole!):

```
lat= lat1+d*cos(tc)
dphi=log(tan(lat/2+pi/4)/tan(lat1/2+pi/4))
IF (abs(lat-lat1) < sqrt(TOL)){
    q=cos(lat1)
} ELSE {
    q= (lat-lat1)/dphi
}
dlon=-d*sin(tc)/q
lon=mod(lon1+dlon+pi,2*pi)-pi
```

TOL is a small number of order machine precision- say 1e-15. The tests avoid 0/0 indeterminacies on E-W courses.

Example:

Suppose point 1 is LAX: (33deg 57min N, 118deg 24min W)  
 Suppose point 2 is JFK: (40deg 38min N, 73deg 47min W)

Rhumb line course from LAX to JFK: LAX (0.592539,2.066470) and JFK is (0.709185,1.287762)

```
dlon_W=mod(1.287762-2.066470,2*pi)=5.504478
dlon_E=mod(2.066470-1.287762,2*pi)=0.778708

dphi=log(tan(0.709185/2+pi/4)/tan(0.592539/2+pi/4))
    =0.146801
q= (0.709185-0.592539)/0.146801 =0.794586
    dlon_E < dlon_W: East is shorter!
tc=mod(atan2(0.778708,0.146801),2*pi)= 1.384464 radians = 79.32 degrees
d=sqrt(0.794586^2*0.778708^2 + (0.709185-0.592539)^2)
= 0.629650 radians = 2164.6 nm
```

Compare this with the great circle course of 66 degrees and distance of 2144 nm.

Conversely, if we proceed 2164.6nm (0.629650 radians) on a rhumbline course of 79.3 degrees (1.384464 radians) starting at LAX, our final point will be given by:

```
lat=0.592539 + 0.629650 * cos(1.384464)
    = 0.709185
dphi=log(tan(0.709185/2+pi/4)/tan(0.592539/2+pi/4))
    =0.146801
q= (0.709185-0.592539)/0.146801 =0.794586
dlon=-0.629650*sin(1.384464)/0.794586=-0.778708
lon=mod(2.066470-0.778708+pi,2*pi)-pi
    =1.287762
```

which is the lat/lon of JFK- as required.

## Local, flat earth approximation

If you stay in the vicinity of a given fixed point (lat0,lon0), it may be a good enough approximation to consider

the earth as "flat", and use a North, East, Down rectangular coordinate system with origin at the fixed point. If we call the changes in latitude and longitude  $dlat=lat-lat_0$ ,  $dlon=lon-lon_0$  (Here treating North and East as positive!), then

```
distance_North=R1*dlat
distance_East=R2*cos(lat0)*dlon
```

where  $R_1$  and  $R_2$  are both the radius of the earth in a spherical model. Taking a spheroidal model (like the WGS84 of GPS)

```
R1=a*(1-e^2)/(1-e^2*(sin(lat0))^2)^(3/2)
R2=a/sqrt(1-e^2*(sin(lat0))^2)
```

$a$  is the equatorial radius of the earth ( $=6378.137000\text{km}$  for WGS84), and  $e^2=2*f*(1-f)$  with the flattening  $f=1/298.257223563$  for WGS84.

$R_1$  and  $R_2$  are called the meridional radius of curvature and the radius of curvature in the prime vertical, respectively.

These approximations fail in the vicinity of either pole.

## Wind Triangles

In all formulae, all angles are in radians. Convert back and forth as in the Great Circle section. [This is unnecessary on calculators which have a "degree mode" for trig functions. Most programming languages provide only "radian mode".]

```
angle_radians=(pi/180)*angle_degrees
angle_degrees=(180/pi)*angle_radians
```

A further conversion is required if using degrees/minutes/seconds:

```
angle_degrees=degrees+(minutes/60.)+(seconds/3600.)

degrees=int(angle_degrees)
minutes=int(60*(angle_degrees-degrees))
seconds=60*(60*(angle_degrees-degrees)-minutes)
```

[ You may have a built-in HH <-> HH:MM:SS conversion to do this efficiently]

Let CRS=course, HD=heading, WD=wind direction (from), TAS=True airspeed, GS=groundspeed, WS=windspeed.

Units of the speeds do not matter as long as they are all the same.

(1) Unknown Wind:

```
WS=sqrt( (TAS-GS)^2+ 4*TAS*GS*(sin((HD-CRS)/2))^2 )
WD=CRS + atan2(TAS*sin(HD-CRS), TAS*cos(HD-CRS)-GS)  (**)
IF (WD<0) THEN WD=WD+2*pi
IF (WD>2*pi) THEN WD=WD-2*pi
( (**) assumes atan2(y,x), reverse arguments if your implementation
has atan2(x,y) )
```

(2) Find HD, GS

```
SWC=(WS/TAS)*sin(WD-CRS)
IF (abs(SWC)>1)
    "course cannot be flown-- wind too strong"
ELSE
    HD=CRS+asin(SWC)
    if (HD<0) HD=HD+2*pi
    if (HD>2*pi) HD=HD-2*pi
    GS=TAS*sqrt(1-SWC^2)-WS*cos(WD-CRS)
ENDIF
```

Note:

The purpose of the "if (HD<0) HD=HD+2\*pi; if (HD>2\*pi) HD=HD-2\*pi" is to ensure the final heading ends up in the range (0, 2\*pi). Another way to do this, with the MOD function available is:

```
HD=MOD(HD, 2*pi)
```

(3) Find CRS, GS

```
GS=sqrt(WS^2 + TAS^2 - 2*WS*TAS*cos(HD-WD))
WCA=atan2(WS*sin(HD-WD), TAS-WS*cos(HD-WD)) (*)
CRS=MOD(HD+WCA, 2*pi)
```

(\*)  $WCA = \arcsin((WS/GS) * \sin(HD-WD))$  works if the wind correction angle is less than 90 degrees, which will always be the case if  $WS < TAS$ . The listed formula works in the general case

### Head- and cross-wind components.

```
HW= WS*cos(WD-RD)      (tailwind negative)
XW= WS*sin(WD-RD)      (positive= wind from right)
```

where HW, XW, WS are the headwind, crosswind and wind speed. WD and RD are the wind direction (from) and runway direction.

As usual, unless you have a version of sin and cos available that takes degree arguments, you'll need to convert to radians.

Example: Wind 060 @ 20 departing Runway 3.

```
WS=20 knots
WD=60 degrees = 60*pi/180 radians
RD=30 degrees = 30*pi/180 radians
Plugging in:
Headwind=17.32 knots
Crosswind = 10 knots (from right)
```

---

### TAS and windspeed from three (GPS) groundspeeds.

Determine your groundspeed on three headings that differ by 120 degrees (eg 40, 160 and 280 degrees), call these  $v_1$ ,  $v_2$  and  $v_3$

Let

```

vms = (v1^2 + v2^2 + v3^2) / 3
a1= v1^2/vms -1
a2= v2^2/vms -1
a3= v3^2/vms -1
mu= (a1^2 + a2^2 + a3^2) / 6

```

Let  $b_p$  and  $b_m$  be the roots of the quadratic  $b^2 - b + \mu = 0$  ie:

```

bp= 1/2 +sqrt(1/4-mu)
bm= mu/bp

```

The TAS and windspeed are then given by  $\sqrt{vms*b_p}$  and  $\sqrt{vms*b_m}$  provided that the TAS exceeds the windspeed. If this is not the case, the roots are exchanged. This is a handy way to check your TAS (and the calibration of your airspeed indicator) using your GPS groundspeed, even though the wind is unknown.

---

### Approximate variation formulae.

I did a least squares polynomial fit to the NFDC airport database.

```

x=latitude (N degrees) y=longitude (W degrees) var= variation (degrees)

var= -65.6811 + 0.99*x + 0.0128899*x^2 - 0.0000905928*x^3 + 2.87622*y -
      0.0116268*x*y - 0.00000603925*x^2*y - 0.0389806*y^2 -
      0.0000403488*x*y^2 + 0.000168556*y^3

```

Continental US only, 3771 points, RMS error 1 degree All within 2 degrees except for the following airports: MO49 MO86 MO50 3K6 02K and KOOA

```

(24 < x < 50, 66 < y < 125)
-----
Alaska Fit, better than 1 degree, all points:
var= 618.854 + 2.76049*x - 0.556206*x^2 + 0.00251582*x^3 - 12.7974*y +
      0.408161*x*y + 0.000434097*x^2*y - 0.00602173*y^2 -
      0.00144712*x*y^2 + 0.000222521*y^3

55 points (x > 54, 130 < y < 172)

```

For Western Europe, fitting to the 1997 IGRF reference field:

```

var =10.4768771667158 -0.507385322418858*lon +0.00753170031703826*lon^2-
1.40596203924748e-05*lon^3 -0.535560699962353*lat +
0.0154348808069955*lat*lon -8.07756425110592e-05*lat*lon^2 +
0.00976887198864442*lat^2 -0.000259163929798334*lat^2*lon-
3.69056939266123e-05*lat^3;

```

Here \*East\* lon is positive! In the range  $-10 < \text{lon} < 28$ ,  $36 < \text{lat} < 68$  RMS error = 0.04 degrees, max error 0.20 degrees.

I've written software that computes magnetic variation anywhere on (or above) the earth's surface, using either the WMM or IGRF reference models. There are [Mac](#), [DOS](#) and [Linux](#) executables available. My web calculator is [here](#)



## Standard Atmosphere and Altimetry

The following contains some formulae concerning altimetry and the standard atmosphere (1976 International Standard Atmosphere).

At sea-level on a standard day:

the temperature,  $T_0 = 59^{\circ}\text{F} = 15^{\circ}\text{C} = 288.15\text{K}$  (C=Celsius K=Kelvin,  
 $T(\text{Kelvin})=T(\text{Celsius})+273.15$ )

the pressure,  $P_0 = 29.92126 \text{ "Hg} = 1013.250 \text{ mB} = 2116.2166 \text{ lbs/ft}^2$   
 $= 760.0 \text{ mmHg} = 101325.0 \text{ Pa} = 14.69595 \text{ psi} = 1.0 \text{ atm}$

the air density,  $\rho_0 = 1.2250 \text{ kg/m}^3 = 0.002376892 \text{ slugs/ft}^3$

The standard lapse rate is  $T_r = 0.0065^{\circ}\text{C/m} = .0019812^{\circ}\text{C/ft}$  below the tropopause  $h_{Tr} = 11.0\text{km} = 36089.24\text{ft}$

Above the tropopause, standard temperature is  $T_{Tr} = -56.5^{\circ}\text{C} = 216.65\text{K}$  (up to an altitude of 20km) Standard temperature at altitude  $h$  is thus given by:

$$\begin{aligned} T_s &= T_0 - T_r \cdot h & (h < h_{Tr}) \\ &= T_{Tr} & (h > h_{Tr}) \\ &= 15 - .0019812 \cdot h(\text{ft}) \text{ } ^{\circ}\text{C} & (h < 36089.24\text{ft}) \end{aligned}$$

Variation of pressure with altitude:

$$\begin{aligned} p &= P_0 \cdot (1 - 6.8755856 \cdot 10^{-6} h)^{5.2558797} & h < 36,089.24\text{ft} \\ p_{Tr} &= 0.2233609 \cdot P_0 \\ p &= p_{Tr} \cdot \exp(-4.806346 \cdot 10^{-5} (h - 36089.24)) & h > 36,089.24\text{ft} \end{aligned}$$

Variation of density with altitude:

$$\begin{aligned} \rho &= \rho_0 \cdot (1 - 6.8755856 \cdot 10^{-6} h)^{4.2558797} & h < 36,089.24\text{ft} \\ \rho_{Tr} &= 0.2970756 \cdot \rho_0 \\ \rho &= \rho_{Tr} \cdot \exp(-4.806346 \cdot 10^{-5} (h - 36089.24)) & h > 36,089.24\text{ft} \end{aligned}$$

Relationship of pressure and indicated altitude:

$$\begin{aligned} &\text{alt\_set in inches, heights in feet} \\ P_{alt\_corr} &= 145442.2 \cdot (1 - (\text{alt\_set}/29.92126)^{0.190261}) \quad \text{or} \\ P_{alt\_corr} &= (29.92 - \text{alt\_set}) \cdot 1000 \quad (\text{simple approximation}) \\ P_{alt} &= \text{Ind\_Alt} + P_{alt\_corr} \end{aligned}$$

Relationship of pressure and density altitude:

$$D\_Alt = P\_alt + (T\_s / T\_r) * (1 - (T\_s / T)^{0.2349690})$$

(Standard temp  $T\_s$  and actual temp  $T$  in Kelvin)

An approximate, but fairly accurate formula is:

$$D\_Alt = P\_Alt + 118.6 * (T - T\_s)$$

where  $T$  and  $T\_s$  may (both) be either Celsius or Kelvin

Density altitude example:

Let pressure altitude ( $P\_alt$ ) be 8000 ft, temperature 18C.

Standard temp ( $T\_s$ ) is given by

$$T\_s = 15 - .0019812 * 8000 = -0.85C = (273.15 - 0.85)K = 272.30K$$

Actual temperature ( $T$ ) is

$$18C = (273.15 + 18)K = 291.15K$$

$$\begin{aligned} \text{Density altitude } (D\_Alt) &= 8000 + (272.30 / .0019812) * (1 - (272.30 / 291.15)^{0.2349690}) \\ &= 8000 + 2145 = 10145ft \end{aligned}$$

or approximately:

$$\text{Density Altitude} = 8000 + 118.6 * (18 + 0.85) = 10236ft$$

Relationship of true and calibrated (indicated) altitude:

$$TA = CA + (CA - FE) * (ISADEV) / (273 + OAT)$$

where

TA= True Altitude above sea-level

FE= Field Elevation of station providing the altimeter setting

CA= Calibrated altitude= Altitude indicated by altimeter when set to the altimeter setting, corrected for calibration error.

ISADEV= Average deviation from standard temperature from standard in the air column between the station and the aircraft (in C)

OAT= Outside air temperature (at altitude)

The above is more precise than provided by the E6B or similar.

**Mach numbers, true vs calibrated airspeeds etc.**

$$\text{Mach Number } (M) = TAS / CS$$

$$CS = \text{sound speed} = 38.967854 * \sqrt{T + 273.15} \quad \text{where } T \text{ is the OAT in celsius.}$$

TAS is true airspeed in knots.

Because of compressibility, the measured IAT (indicated air temperature) is higher than the actual true OAT. Approximately:

$$IAT = OAT + K * TAS^2 / 7592$$

The recovery factor K, depends on installation, and is usually in the range 0.95 to 1.0, but can be as low as 0.7. Temperatures are Celsius, TAS in knots.

Also:

$$OAT = (IAT + 273.15) / (1 + 0.2 * K * M^2) - 273.15$$

The airspeed indicator measures the differential pressure, DP, between the pitot tube and the static port, the resulting indicated airspeed (IAS), when corrected for calibration and installation error is called "calibrated airspeed" (CAS).

For low-speed ( $M < 0.3$ ) airplanes the true airspeed can be obtained from CAS and the density altitude, DA.

$$TAS = CAS * (\rho_0 / \rho)^{0.5} = CAS / (1 - 6.8755856 * 10^{-6} * DA)^{2.127940} \quad (DA < 36,089.24 \text{ ft})$$

Roughly, TAS increases by 1.5% per 1000ft.

When compressibility is taken into account, the calculation of the TAS is more elaborate:

$$\begin{aligned} DP &= P_0 * ((1 + 0.2 * (IAS / CS_0)^2)^{3.5} - 1) \\ M &= (5 * ((DP / P + 1)^{(2/7)} - 1))^{0.5} \quad (*) \\ TAS &= M * CS \end{aligned}$$

[(\*) If this results in  $M > 1$  - ie supersonic flight, we have to account for the shock wave ahead of the pitot tube, using [Rayleigh's Supersonic Pitot equation](#).

Using the M from above as the first guess on the RHS, iterate:

$$M = 0.881285 \sqrt{((DP / P + 1) (1 - 1 / (7M^2))^{(5/2)})}$$

to convergence.]

$P_0$  is is (standard) sea-level pressure,  $CS_0$  is the speed of sound at sea-level, CS is the speed of sound at altitude, and P is the pressure at altitude.

These are given by earlier formulae:

$$\begin{aligned} P_0 &= 29.92126 \text{ "Hg} = 1013.25 \text{ mB} = 2116.2166 \text{ lbs/ft}^2 \\ P &= P_0 * (1 - 6.8755856 * 10^{-6} * PA)^{5.2558797}, \text{ pressure altitude, } PA < 36,089.24 \text{ ft} \\ CS &= 38.967854 * \sqrt{T + 273.15} \quad \text{where T is the (static/true) OAT in Celsius.} \\ CS_0 &= 38.967854 * \sqrt{15 + 273.15} = 661.4786 \text{ knots} \end{aligned}$$

$$\begin{aligned} \text{[Example: CAS=250 knots, PA=10000ft, IAT=2C, recovery factor=0.8} \\ DP &= 29.92126 * ((1 + 0.2 * (250 / 661.4786)^2)^{3.5} - 1) = 3.1001 \text{ "} \\ P &= 29.92126 * (1 - 6.8755856 * 10^{-6} * 10000)^{5.2558797} = 20.577 \text{ "} \\ M &= (5 * ((3.1001 / 20.577 + 1)^{(2/7)} - 1))^{0.5} = 0.4523 \text{ Mach} \\ OAT &= (2 + 273.15) / (1 + 0.2 * 0.8 * 0.4523^2) - 273.15 = -6.72 \text{ C} \\ CS &= 38.967854 * \sqrt{-6.7 + 273.15} = 636.08 \text{ knots} \end{aligned}$$

$$TAS = 636.08 * 0.4523 = 287.7 \text{ knots}$$

In the reverse direction, given Mach number M and pressure altitude PA, we can find the IAS with:

$$x = (1 - 6.8755856e-6 * PA)^{5.2558797}$$

$$ias = 661.4786 * (5 * ((1 + x * ((1 + M^2/5)^{3.5} - 1))^{(2/7.)} - 1))^{0.5} \quad (\text{for } M \leq 1)$$


---

Some notes on the origins of some of the "magic" number constants in the preceeding section:

$6.8755856 * 10^{-6} = T'/T_0$ , where T' is the standard temperature lapse rate and T<sub>0</sub> is the standard sea-level temperature.

$5.2558797 = Mg/RT'$ , where M is the (average) molecular weight of air, g is the acceleration of gravity and R is the gas constant.

0.2233609 = ratio of the pressure at the tropopause to sea-level pressure.

$4.806346 * 10^{-5} = Mg/RT_{tr}$ , where T<sub>tr</sub> is the temperature at the tropopause.

$4.2558797 = Mg/RT' - 1$

0.2970756 = ratio of the density at the tropopause to the density at SL (rho<sub>0</sub>)

$145442 = T_0/T'$

$38.967854 = \sqrt{\gamma R/M}$  (in knots/Kelvin<sup>0.5</sup>), where gamma is the ratio of the specific heats of air

---

## Relative humidity, dewpoint, frostpoint etc.

The relative humidity, f (as a fraction) is related to the temperature, T and dewpoint Td by:

$$f = \exp(17.27 (Td / (Td + 237.3) - T / (T + 237.3)))$$

and to the frostpoint temperature Tf by:

$$f = \exp(21.87 (Tf / (Tf + 265.5) - T / (T + 265.5)))$$

Temperatures are in Celsius. Multiply f by 100 if you want a percentage. The above are based on an empirical fit to the saturation vapor pressure of water due to O. Tetens in *Zeitschrift für Geophysik, Vol VI* (1930), quoted in *"Principles of Meteorological Analysis"* by W. J. Saucier (Dover NY 1983).

This fit is:

$$e_s = 6.11 * \exp(bT / (T + a)) \quad \text{for the saturation vapor pressure } e_s \text{ in mbar}$$

over water a=237.3, b=17.27  
over ice a=265.5, b=21.87

An alternative slightly more accurate fit (over water) is:

$$e_s = 6.10779 + T * (4.43652e-1 + T * (1.42894e-2 + T * (2.65064e-4 + T * (3.03124e-6 + T * (2.03408e-8 + (6.13682e-11 * T))))))$$

(from Lowe, JAM (1977), 103)

The latest and greatest [fit](#) based on "Thermodynamic properties of Dry Air, Moist Air and Water, and SI Psychrometric charts" by Arnold Wexler and Richard Hyland, National Bureau of Standards; Richard Stewart, University of Idaho.

Tables of Relative Humidity and Dewpoint vs Temperature and Wet Bulb Temperature can be found in "Introduction to Meteorology" by Franklyn Cole (Wiley NY 1975).

Inverting this to find dewpoint in terms of temp and RH:

```
Dewpoint    Td=237.3/(1/(ln(f)/17.27+T/(T+237.3))-1)
Frostpoint  Tf=265.5/(1/(ln(f)/21.87+T/(T+265.5))-1)
```

Given the wet bulb temperature Tw (C), the dry bulb temperature T (C), and the pressure, p in mbar one gets the (approximate) relative humidity and dewpoint by the following:

```
ed= 6.11*exp(17.27*T/(T+237.3))      /* SVP at dry-bulb temp
ew= 6.11*exp(17.27*Tw/(Tw+237.3))    /* SVP at wet-bulb temp
wd=0.62197*ed/(p-ed)                 /* saturation mixing ratio at T
ww=0.62197*ew/(p-ew)                 /* saturation mixing ratio at Tw
w=(2500.0*ww-1.0046*(T-Tw))/(2500.0+1.81*(T-Tw)) /* mixing ratio
f= w/wd                               /* relative humidity as a fraction
e= p*w/(0.62197+w)                   /* vapor pressure (mb)
Td=(237.3*log10(e)-186.527)/(8.286-log10(e)) /* the dewpoint (C)
```

This uses the Tetens fit for the saturated vapor pressure and treat water vapor as an ideal gas, both of which are pretty good approximations. If you want better refer to the Smithsonian Meteorological Tables ( *Smithsonian Institute 1963* )

---

A related formula gives the increase in effective density altitude due to humidity. It only addresses the reduction of air density, and not the effect on engine power output:

```
Increase(ft)=0.267*RH*(T+273)*exp(17.3*T/(T+237))*(1-0.00000688*H)^(-5.26)
```

RH (f above) is the relative humidity expressed as a fraction, T is the temperature in Celsius and H is the pressure altitude in feet.

Examples are:

```
SL/30C/100% -> 565' increase in DA
10000/5C/80% -> 124' increase in DA
5000/40C/80% -> 977' increase in DA.
```

In terms of the dewpoint, Td the formula is:

```
Increase(ft)=0.267*(T+273)*exp(17.3*Td/(Td+237))*(1-0.00000688*H)^(-5.26)
```

which clearly agrees with the above when T=Td and RH=1.

---

**Bellamy's formula.**

Bellamy's formula for the wind drift and (single) wind correction angle is as follows:

$$\begin{aligned}\text{Drift (nm)} &= 21500 * (p2-p1) / (\sin(\text{latitude}) * \text{TAS}) && (p2-p1 \text{ in inches}) \\ &= 635 * (p2-p1) / (\sin(\text{latitude}) * \text{TAS}) && (p2-p1 \text{ in mB})\end{aligned}$$

$$\begin{aligned}\text{Wind Correction Angle} &= 1230000 * (p2-p1) / (\sin(\text{latitude}) * \text{TAS} * \text{Dist}) && (\text{inches}) \\ &= 36300 * (p2-p1) / (\sin(\text{latitude}) * \text{TAS} * \text{Dist}) && (\text{mB})\end{aligned}$$

$p2-p1$  is the difference between the destination and departure pressures.  $\text{latitude}$  is the average latitude on the route.  $\text{TAS}$  is the true airspeed in knots.  $\text{Dist}$  is the distance in nm.

If the destination pressure is higher, the drift is to the left, and the required WCA is to the right (and vice-versa).

Example:

SFO -> LAX 300nm at 100 knots, latitude 36 degrees. Suppose the LAX altimeter setting is 0.2" higher (better the actual pressure difference at cruise altitude if you can get it).

$$\begin{aligned}\text{Drift} &= 21500 * 0.2 / (\sin(36) * 100) = 73\text{nm left} \\ \text{WCA} &= 1230000 * 0.2 / (\sin(36) * 100 * 300) = 14 \text{ degrees right}\end{aligned}$$

A discussion of this is in Barry Schiff's "*Proficient Pilot I*".

Unit conversions, etc.

1 knot = 1.150779 mph  
 1 mph = 0.868976 knot  
 1 knot = 1.852000 km/hr\*  
 1 km/hr = 0.539968 knot  
 1 mph = 1.609344 km/hr\*  
 1 km/hr = 0.621371 mph

\* = exact conversion factor

Ellipsoidal parameters:

Name	Major axis, a (km)	Flattening (f)
WGS84	6378.13700	1/298.257223563
GRS80/NAD83	6378.13700	1/298.257222101
WGS66	6378.145	1/298.25
GRS67/IAU68	6378.16000	1/298.2472
WGS72	6378.135	1/298.26
Krasovsky	6378.245	1/298.3
Clarke66/NAD27	6378.2064	1/294.9786982138

Reference: *Coordinate Systems and Map Projections*, D. H. Maling (Pergamon 1992) (except Clarke66 !)

To convert between geocentric (radius  $r$ , geocentric latitude  $u$ ) and geodetic coordinates (geodetic latitude  $v$ , height above the ellipsoid  $h$ ):

$$\tan(u) = \tan(v) * (h * \sqrt{(a * \cos(v))^2 + (b * \sin(v))^2} + b^2) / (h * \sqrt{(a * \cos(v))^2 + (b * \sin(v))^2} + a^2)$$

$$r^2 = h^2 + 2h \sqrt{(a \cos(v))^2 + (b \sin(v))^2} + \frac{(a^4 - (a^4 - b^4) \sin^2(v))}{(a^2 - (a^2 - b^2) \sin^2(v))}$$

$a$  and  $b$  are the semi-major axes of the ellipsoid, and  $b=a*(1-f)$ , where  $f$  is the flattening. Note that geocentric and geodetic longitudes are equal.

---

## Turns and pivotal altitude

In a steady turn, in no wind, with bank angle,  $b$  at an airspeed  $v$

$$\tan(b) = v^2 / (R g)$$

$$v = w R$$

where  $g$  is the acceleration due to gravity,  $R$  is the radius of turn and  $w$  is the rate of turn.

Pivotal altitude  $h_p$  is given by

$$h = v^2 / g$$

With  $R$  in feet,  $v$  in knots,  $b$  in degrees and  $w$  in degrees/sec (inconsistent units!), numerical constants are introduced:

$$R = v^2 / (11.23 \tan(0.01745 * b))$$

(Example) At 100 knots, with a 45 degree bank, the radius of turn is  $100^2 / (11.23 \tan(0.01745 * 45)) = 891$  feet.

The rate of turn  $w$  is given by:

$$w = 96.7 * v / R$$

(Example)  $= 96.7 * 100 / 891 = 10.9$  degs/sec

The bank angle  $b_s$  for a standard rate turn is given by:

$$b_s = 57.3 * \arctan(v / 362.1)$$

(Example) for 100 knots,  $b_s = 57.3 * \arctan(100 / 362.1) = 15.4$  degrees

A useful rule-of-thumb, accurate to ~1 degree for speeds less than 250 knots, is  $b_s = v / 7$  ( $v$  in knots).

The pivotal altitude is given by:

$$h_p = v^2 / 11.23$$

(Example) At 100 knots groundspeed the pivotal altitude is  $100^2 / 11.23 = 890$  feet.

---

## Distance to horizon

At a height  $h$  above the ground, the distance to the horizon  $d$ , is given by:

$$d = \sqrt{2Rh/b}$$

$b=0.8279$  is a factor that accounts for atmospheric refraction and depends on the atmospheric temperature lapse rate, which is taken to be standard.  $R$  is the radius of the earth. Note that the earth is assumed smooth- likely only true over the oceans!

For  $h$  in feet and  $d$  in nm:

$$d = 1.17 \sqrt{h}$$

i.e. from 10000 feet, the horizon is 117nm away

(Reference [Bowditch](#) American Practical Navigator (1995) Table 12.)

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## Revision History

*Version 1.37 11/6/02*

Added note on choice of earth's radius.

1.36/pre>

Added section on local, flat earth approximation.  
Treated supersonic case of Mach/CAS relationship

1.35

Corrected typo  $\text{mod}(y,x) = y - x \cdot \text{floor}(y/x)$

1.34

Added section on along-track distance. Made lat/lon given radial and distance numerical example more complete.

1.33

Changed section added in 1.32 to interpolate distances along the great circle connecting the points, not the chord, which is more useful and what was originally intended!

1.32

Added section on intermediate points between two others - given the fractional distance along the GC route between them.

1.31

Added section on distance to horizon



1.30

Courses starting at the S. Pole now 360 degrees not 0.  
Added reference to sample spreadsheet

1.29

Added formulae to determine TAS and windspeed from groundspeeds on three headings that differ by 120 degrees ie that form an equilateral triangle.

Corrected bug in "Heron" formula for spherical excess.

1.28

Added implementations of asin and acos with only atan available.  
(For those crippled with Visual Basic)

1.27

$4.2558797 = Mg/RT'^{-1}$  not  $Mg/RT_0^{-1}$  , and

$38.967854 = \sqrt{\gamma R/M}$  (in knots/Kelvin<sup>0.5</sup>) (numerical formulae OK)

Added reference to intersections of two great circles.

1.26

$5.2558797 = Mg/RT'$  not  $Mg/RT_0$  (numerical formulae OK)

1.25

Head and cross-wind components.

1.24

Corrected some last digit rounding errors in the rhumb line examples. Added a reference to the latest (1983) National Bureau of Standards fits to the vapor pressure of water over water and ice.  
(Thank you Oscar Van Vlijmen!)

1.23

Additions to spherical triangle section

1.22

Course between points formula failed if the initial point was exactly a pole. This has to be special-cased.

1.21

Added references to sunrise/sunset

Added Mach -> IAS formulae

1.20

Third numerical example of the effect of humidity on density altitude corrected.

Added standard rate turn bank angle rule-of-thumb.

1.19

Another bug in the intersection section... The test for input data where "no intersection exists", or more precisely, when it's ambiguous which of the two great circle intersections is desired, was misplaced. With valid data, no problem...

The Clarke66/NAD27 inverse flattening was incorrect in my reference book. Corrected.

Thanks to Larry Lewis.

1.18

Corrected equation for dst12 in intersection section. (Should have been the distance formula in the first section!) The numerical example used the correct formula.

1.17

(1/26/98) Changed formula constants to use 1976 US/ICAO Standard Atmosphere instead of 1962 US Standard. Made unit conversions more accurate.

(by Doug Haluza)

1.16

(10/26/97) Corrected conversion to hh:mm:ss  
 $\text{seconds} = 60 * (60 * (\text{angle\_degrees} - \text{degrees}) - \text{minutes})$

1.15

(9/11/97) Added European variation fit

1.14

(9/2/97) Added warnings about arguments of asin and acos being out of range from rounding error.

1.13

(8/31/97) The rhumb line section was rewritten. Erroneously corrected one formula, then changed it back! Added a numerical example for the calculation of the endpoint of a rhumb line. Added some more spherical triangle formulae.

1.12

Somehow I dropped a line in the the 1.08 atan2 fix. Sigh!  
Added turn radius, pivotal altitude formulae.

1.11

Made "Lat/lon given radial and distance" handle the pole endpoint case more elegantly.

1.10

Add "find CRS, GS" to wind triangle section

1.09

Added geodetic/geocentric coordinate conversion

1.08

Added an alternative method for calculation of course between two points, not requiring pre-computation of the distance between them.

Changed the definition of atan2 to the ANSI standard one where it is defined to have a range of  $-\pi < \text{atan2} \leq \pi$ , rather than  $0 \leq \text{atan2} < 2\pi$ . This was a bug only if had you used the previous version to define asin in terms of atan via atan2. No one reported it though...

Corrected some damaged formulae in the intersection section of the html version.

1.07 (4/1/97)

Add additional spherical triangle formulae. Correct the condition ( $d\text{lon} < \pi/2$ ) for the validity of the short range formula in the "lat/lon given radial and distance" section.

1.06 (3/3/97)

Correct typo in html version of HDG/GS formula. (minus sign)  
Definitions of a and b swapped in Tejen's fit to saturation vapor pressure.

1.05 (12/17/96)

Correct test for pole in formula for computing lat/long of a point a given radial and distance:  $\text{lat}=0 \Rightarrow \cos(\text{lat})=0$

1.04 (11/11/95)

Add formula for computing lat/long of a point a given radial and distance valid when the distance can exceed one quarter of the earth's circumference.

Note that atan2(0,0) should return an error.

Add rhumb line formulae and example.

Change intersection calculation to only provide result when intersection of radials exists.

Comments, corrections, suggestions to:

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